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A New Proof of Ruelle's Superstability Bounds

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For a slightly stronger assumptions on the interaction we give a very transparent proof of Ruelle's result in the language of Poisson integral measure representation for the correlation functions on the configuration space using some kind of cluster expansion in the densities of configurations.

KEY WORDS: Thermodynamic limit; continuous classical system; superstability estimates; Poisson measure.

1. INTRODUCTION

Proof of the existence of the correlation functions of classical statistical mechanics in thermodynamic limit at arbitrary nonnegative values of activity z and inverse temperature $\beta = (kT)^{-1}$ follows from their uniform (in volume Λ) bounds:

$$\rho^{\Lambda}(x_1,..., x_m) \leqslant \xi^m \tag{1.1}$$

This inequality was proved by D. Ruelle[1] in 1970 for pair, superstable and lower regular potentials. It allowed to extend various results obtained by R. L. Dobrushin and R. A. Minlos[2] for the case of potentials, which were non-integrably divergent at the origin. The proof was based on a careful analysis of the configurations of particles in Λ and partition of these configurations into subsets, taking into account maximum of the particles in bounded regions. Moreover the superstable condition allowed to show that Gibbs factor for the configurations with charge number of particles in a small volume decreased with a number of particles. But the proof of the inequality (1.1) was based on a large number of technical lemmas and propositions, which required additional constructions and calculations.

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On the other hand in the series of works[3–6] the representation for correlation functions in a finite volume by Poisson measure integrals on $\mathscr{S}(\mathbb{R}^\nu)$ (or on the space of configurations Γ as a carrier set of this measure) were proposed. However, we should note that the notation of the grand canonical measure (or in physical language ideal gas measure) as

$$\sum_{n\geq 0} \frac{z^n}{n!} \int_{\Lambda^n} dx_1 \cdots dx_n \cdots \equiv \int d\omega_\Lambda \cdots$$

were used long ago by many authors (see for example refs. 7–9), but the properties of this measure and integrals were not used intensively. In the mentioned papers[3–6] the infinite divisible property of the Poisson measure was used to construct cluster expansion and prove their convergence.

From the point of view of the point measure theory the problem of the construction of Gibbs modification of the Poisson free measure in thermodynamic limit on the space of configurations is nontrivial mathematical problem because of the space of configurations is nonlinear space. In the articles by S. Albeverio, Yu. G. Kondratiev and M. Röckner[10–11] analysis and geometry on the configuration space were developed. As a particular result a slight modification of the bounds (1.1) were proved (using (1.1) and Mayer–Montroll equations):

$$\rho^\Lambda(x_1,..., x_m) \leq \xi^m \exp\left\{-\beta \sum_{1\leq i<j\leq m} \varphi(x_i - x_j)\right\} \tag{1.2}$$

where $\varphi(x)$ is 2-body interaction potential. This is more natural behaviour of the correlation functions for small distances between particles (see also ref. 2).

The main goal of this paper is to prove the inequalities (1.1) and (1.2) using Poisson integral representation for the correlation functions without additional constructions. The proof is very simple and transparent and based on the expansion of the Poisson integral over all configurations into the series over some kind "dense" configurations. In fact the Ruelle's idea is realized in a very natural way in the language of Poisson measure integrals over the space of configurations.

Nevertheless, we should note that we use a little bit stronger conditions on the potential than in the original work by D. Ruelle.[1]

A short contents of this article is the following. In Section 2 we give some notations, define the system and formulate the main result. In Section 3 we construct Poisson integral representation and expansion over densities of the configurations. In Section 4 we give all needed estimates to prove the main theorem.

2. DEFINITIONS AND MAIN RESULT

2.1. Some Notations

1^0. Euclidean ν-dimensional space: $\mathbb{R}^\nu$.

2^0. Lattice in $\mathbb{R}^\nu$ with a cell of length λ: $\lambda\mathbb{Z}^\nu$.

3^0. Elementary cubs with ribs λ, centered in $r \in \mathbb{Z}^\nu$:

$$\Delta \equiv Q_r(\lambda) = \{x \in \mathbb{R}^\nu \mid \lambda(r^i - 1/2) \leqslant x^i < \lambda(r^i + 1/2), i = 1, ..., \nu.\}, |\Delta| = \lambda^\nu$$

4^0. The sequence of sets: $(X)_n = (X_1, ..., X_n)$, $X_j \neq \emptyset$, $j = 1, ..., n$.

5^0. The set of sets: $\{X\}_n = \{X_1, ..., X_n\}$.

6^0. R-disjoint set of sets:

$$\{X\}_n^R = \{X_1, ..., X_n \mid \text{dist}(X_i, X_j) > R, i \neq j\}$$

7^0. The union of sets: $\check{X}_n = \bigcup_{j=1}^n X_j$.

8^0. The complementary set: $X^c = \Lambda \backslash X$.

2.2. Definition of the System. Correlation Functions

We consider an infinite system of the classical statistical mechanics of the point, identical particles with 2-body interaction, i.e., the energy of the interaction of N-particles is given by the formula

$$V(x)_N = \sum_{1 \leqslant i < j \leqslant N} \varphi(x_i - x_j)$$

where $\varphi(x) = \varphi(|x|)$ is the interaction potential. The N-particle energy satisfies the following condition:

$$V(x)_N \geqslant \sum_{r \in \mathbb{Z}^\nu} (AN_r^2 - BN_r), \quad N = \sum_{r \in \mathbb{Z}^\nu} N_r \tag{2.1}$$

where N_r is the number of particles in an elementary cube Δ. If $A = 0$, $B \geqslant 0$, the condition (2.1) is called *stability condition*; if $A > 0$, $B \geqslant 0$, the Eq. (2.1) is called *superstability condition.*

The correlation functions are defined (in a finite volume $\Lambda \subset \mathbb{R}^\nu$) by (see for example refs. 12–13):

$$\rho^\Lambda(x)_m = Z_\Lambda^{-1} \sum_{n \geqslant 0} \frac{z^{n+m}}{n!} \int_{\Lambda^n} dx_{m+1} \cdots dx_{m+n} e^{-\beta V(x)_{m+n}}, \quad \beta = (kT)^{-1} \tag{2.2}$$

where Z_Λ is the partition function, which is the same series at $m=0$ and z, β are nonnegative parameters (activity and inverse temperature).

In this work we consider Λ and its subsets as unions of elementary cubes Δ. We do not discuss here the problem of boundary conditions, so we (for convenience) consider "empty" boundary conditions.

2.3. Poisson Measure and Configuration Space

Using the definition of the Poisson measure it is easy to rewrite the Eq. (2.2) in the form of integral (see refs. 3–6 for details):

$$\rho^\Lambda(x)_m = Z_\Lambda^{-1} \int_{\Gamma_\Lambda} \pi_z(d\gamma_\Lambda) \exp\{-\beta V(x)_m - \beta V((x)_m; \gamma_\Lambda) - \tfrac{1}{2}\beta V(\gamma_\Lambda)\} \tag{2.3}$$

where

$$\begin{aligned} V((x)_m; \gamma_\Lambda) &= \sum_{j=1}^{m} \int dx\, \varphi(x_j - x)\, \gamma_\Lambda(x) \\ V(\gamma_\Lambda) &= \int dx\, dy{:}\, \gamma_\Lambda(x)\, \varphi(x-y)\, \gamma_\Lambda(y){:} \end{aligned} \tag{2.4}$$

$\pi_z(d\gamma_\Lambda)$ is an unnormalized Poisson measure with the intensity z on the configuration space Γ_Λ, so $\pi_z(\Gamma_\Lambda) = \int_{\Gamma_\Lambda} \pi_z(d\gamma_\Lambda) = e^{z|\Lambda|}$, γ_Λ is the projection of a configuration γ in $\mathbb{R}^\nu$ on Λ. Using Dirac δ-function one can write

$$\gamma_\Lambda(x) = \sum_{j=1}^{|\gamma_\Lambda|} \delta(x - x_j), \qquad x_j \in \Lambda$$

where $|\gamma_\Lambda|$ is cardinality of given configuration $\gamma_\Lambda \equiv \{x_1, ..., x_{|\gamma_\Lambda|}\}$. Wick product in the Eq. (2.4) means subtraction of the diagonal elements $(x=y)$. One can consider the formula (2.3) as a convenient notation of the definition (2.2), but we will use one important property of the Poisson measure—the infinite divisible property, i.e., if we have two functions $F_1(\gamma_{X_1})$ and $F_2(\gamma_{X_2})$ and $X_1 \cap X_2 = \varnothing$, $X_1, X_2 \subset \Lambda$, then (see refs. 4–6 for details):

$$\begin{aligned} &\int_{\Gamma_\Lambda} \pi_z(d\gamma_\Lambda)\, F_1(\gamma_{X_1})\, F_2(\gamma_{X_2}) \\ &\quad = e^{z|(X_1 \cup X_2)^c|} \int_{\Gamma_{X_1}} \pi_z(d\gamma_{X_1})\, F_1(\gamma_{X_1}) \int_{\Gamma_{X_2}} \pi_z(d\gamma_{X_2})\, F_2(\gamma_{X_2}) \end{aligned} \tag{2.5}$$

The exponent in the Eq. (2.5) is due to we consider unnormalized Poisson measure. Besides the *stability condition* in this language is:

$$V_{st}(x)_m + V_{st}((x)_m; \gamma_X) + \tfrac{1}{2} V_{st}(\gamma_X) \geqslant -B(m + |\gamma_X|) \tag{2.6}$$

for any $X \subset \Lambda$ and *stable energy* V.

2.4. Main Result

As we mentioned in the Introduction we consider a slightly stronger assumptions on the interaction potential:

(A1): $\varphi(x) = \varphi_+(x) + \varphi_{st}(x) = \varphi_+(x) + \varphi_{st}^+(x) - \varphi_{st}^-(x)$,
$\varphi_+(x) \geqslant 0$ and $\varphi_+(0)$ is large (see (A3)),
$\varphi_{st}(x)$ satisfies the stability condition (2.6).

(A2): diam supp $\varphi(x) = R < \infty$.

(A3): $\frac{1}{4}b - v_1 - B \geqslant 0$, where $b = \inf_{|x| \leqslant \lambda\sqrt{\nu}} \varphi_+(x)$,

$$v_1 = \sup_{x,\Lambda} \sup_{\gamma_\Lambda^d} \sum_{k=1}^{|\gamma_\Lambda^d|} \varphi_{st}^-(x - x_k) \tag{2.7}$$

where γ_Λ^d is so-called *dilute* configuration, which means that all $x_k, k = 1,..., |\gamma_\Lambda|$ belong to the different elementary cubes $\Delta = Q_r(\lambda)$, i.e., $x_k \in \Delta_k$, $\Delta_j \cap \Delta_k = \varnothing$ for $j \neq k$.

Remarks.

1^0. Even in the case when $\varphi_+(0) < \infty$ (A1) is the sufficient conditions for the potential to be superstable potential. But it is unknown (at least for the author) an example of superstable potential which does not satisfy (A1).

2^0. The assumption (A2) is technical one and we believe that it can be changed by more natural assumption of integrability at large $|x|$.

3^0. The assumption (A3) is very delicate and require rapidly increasing potential at the origin. The sufficient condition for (A3) is nonintegrability $\varphi(x)$ at the origin, because at small λ, $v_1 \sim \lambda^{-\nu} \|\varphi_{st}^-\|_1$, where $\|\cdot\|_1$ is $L_1(\mathbb{R}^\nu)$-norm and $b \sim c_\varphi \lambda^{-\mu}$ in the case when $\varphi_+(x) \sim |x|^{-\mu}$ at the origin. So if $\mu \geqslant \nu$ (and c_φ is large for $\mu = \nu$) the assumption (A3) is true.

Theorem 2.1. For the classical systems, which satisfy the assumptions (A1)–(A3) the correlation functions are bounded from above by:

$$\rho^\Lambda(x)_m \leqslant \xi^m e^{-\beta V_+(x)_m} \tag{2.8}$$

for any β, $z \geqslant 0$, where $\xi = \xi(\beta, z)$ and $V_+(x)_m$ is the energy, constructed by $\varphi_+(x)$.

The proof of the theorem will be done into two steps. First we construct some expansion for $\rho^{\Lambda}(x)_m$ which is an expansion over densities of configurations. The second step is the estimation of Gibbs energy on an every given configuration.

3. CLUSTER EXPANSION IN THE DENSITIES OF CONFIGURATIONS

The main idea of proving Theorem consist in separation dilute configurations from dense configurations. To define this configurations we define indicator function for the configuration in elementary cube Δ:

$$\chi_n^{\Delta}(\gamma_{\Lambda}) = \chi_n^{\Delta}(\gamma_{\Delta}) = \begin{cases} 1, & \text{for } |\gamma_{\Delta}| = n \\ 0, & \text{otherwise} \end{cases}$$

Then the indicator for *dilute* configuration we define as

$$\chi_{-1}^{\Delta}(\gamma_{\Delta}) = \chi_0^{\Delta}(\gamma_{\Delta}) + \chi_1^{\Delta}(\gamma_{\Delta}) \tag{3.1}$$

and for *dense* configuration

$$\chi_{+1}^{\Delta}(\gamma_{\Delta}) = \sum_{n \geqslant 2} \chi_n^{\Delta}(\gamma_{\Delta}) \tag{3.2}$$

To obtain decomposition we use the partition of the unite in the following way

$$1 = \prod_{\Delta \in \Lambda} [\chi_{-1}^{\Delta}(\gamma_{\Delta}) + \chi_{+1}^{\Delta}(\gamma_{\Delta})] = \sum_{\omega} \prod_{\Delta \in \Lambda} \chi_{\omega(\Delta)}^{\Delta}(\gamma_{\Delta}) \tag{3.3}$$

where ω is the map of every elementary cubes of Λ into the numbers $+1$ or -1, $\omega(\Delta) = \pm 1$. Inserting (3.3) into (2 3) we get

$$\rho^{\Lambda}(x)_m = \frac{z^m}{Z_{\Lambda}} e^{-\beta V_+(x)_m} \tilde{\rho}^{\Lambda}(x)_m \tag{3.4}$$

$$\tilde{\rho}^{\Lambda}(x)_m = \sum_{\omega} \int \pi_z(d\gamma_{\Lambda}) \prod_{\Delta \in \Lambda} \chi_{\omega(\Delta)}^{\Delta}(\gamma_{\Delta}) \\ \times \exp\left\{ -\beta V_{st}(x)_m - \beta V((x)_m; \gamma_{\Lambda}) - \frac{1}{2} \beta V(\gamma_{\Lambda}) \right\} \tag{3.5}$$

Now we define the set

$$X \equiv X_+ = \{\Delta \in \Lambda \mid \omega(\Delta) = +1\}$$

Then the sum over all possible ω one can rewrite as the sum over all possible sets $X \equiv X_+$ in Λ. So the Eq. (3.5) is:

$$\tilde{\rho}^{\Lambda}(x)_m = \sum_{\varnothing \subseteq X \subseteq \Lambda} \int \pi_z(d\gamma_\Lambda)\, \tilde{\chi}_+^X(\gamma_X)\, \tilde{\chi}_-^{X^c}(\gamma_{X^c}) \\ \times \exp\{-\beta V_{st}(x)_m - \beta V((x)_m; \gamma_\Lambda) - \tfrac{1}{2}\beta V(\gamma_\Lambda)\} \tag{3.6}$$

Here we use the following notation

$$\tilde{\chi}_\pm^X(\gamma_X) = \prod_{\Delta \in X} \chi_{\pm 1}^{\Delta}(\gamma_\Delta) \tag{3.7}$$

Definition 3.1. For any $X \equiv X_+$ define graph $G_R(X)$ with vertices in the centers of elementary cubes $\Delta \in X$ and lines $l(\Delta, \Delta')$ iff $\mathrm{dist}(\Delta, \Delta') \leqslant R$.

Definition 3.2. The set $X \equiv X_+$ is called R-connected if the correspondent graph $G_R(X)$ is connected in ordinary way.

Then every set $X \equiv X_+$ can be represented as some fixed partition $\{X\}_n^R$ (see notation 6^0.) and so the sum over all possible X in Λ we can change by all possible sets $\{X\}_n^R$ (for $n=0$, $X=\varnothing$). Further we pass from the sum over all such sets to the sum over $X_1,\dots, X_n$ independently and to remove the conditions *dist*$(X_i, X_j) > R$ we introduce *hard-cor* potential

$$\chi_R^{cor}(X)_n = \begin{cases} 0, & \text{if for any } \ X_i, X_j, i \neq j, \mathrm{dist}(X_i, X_j) \leqslant R,\ 1 \leqslant i, j \leqslant n \\ 1, & \text{otherwise} \end{cases}$$

Then we get

$$\tilde{\rho}^{\Lambda}(x)_m = \sum_{n \geqslant 0} \frac{1}{n!} \sum_{(X)_n} \chi_R^{cor}(X)_n \int \pi_z(d\gamma_\Lambda)\, \tilde{\chi}_+^X(\gamma_X)\, \tilde{\chi}_-^{X^c}(\gamma_{X^c}) \\ \times \exp\{-\beta V_{st}(x)_m - \beta V((x)_m; \gamma_\Lambda) - \tfrac{1}{2}\beta V(\gamma_\Lambda)\} \tag{3.8}$$

Now, the last step to arrange our decomposition is the following. Define the set

$$X_R^m = \{\Delta \in \Lambda \mid \mathrm{dist}(\Delta, x_j) \leqslant R,\ j = 1,\dots, m\} \tag{3.9}$$

This is fixed set (for fixed variables of correlation function $\rho_\Lambda(x)_m$). Now we split every sum over X_j, $j=1,\dots,n$ into two sums: the first sum over those X_j, which nonintersect region X_R^m and the second one (we note it by Y_l), which intersect X_R^m. There are $n!/k!\,(n-k)!$ possibilities when any k sets X_j nonintersect X_R^m and $(n-k)$ sets Y_l intersect X_R^m. So the final expansion is the following:

$$\tilde{\rho}^\Lambda(x)_m = \sum_{n\geqslant 0} \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} \sum_{(X)_k \cap X_R^R = \varnothing} \sum_{(Y)_{n-k} \cap X_R^m \neq \varnothing} \chi_R^{cor}((X)_k, (Y)_{n-k})$$
$$\times \int \pi_z(d\gamma_\Lambda)\, \tilde{\chi}_+^X(\gamma_X)\, \tilde{\chi}_-^{X^c}(\gamma_{X^c})$$
$$\times \exp\left\{-\beta V_{st}(x)_m - \beta V((x)_m;\gamma_\Lambda) - \frac{1}{2}\beta V(\gamma_\Lambda)\right\} \tag{3.10}$$

4. PROOF OF THEOREM

The first step of our estimation is to split exponent in (3.10) into three parts, which corresponds the interactions of the particles inside the region $X_R^m \cup \check{Y}_{n-k}$ and the interactions between these particles and all particles of the dilute configuration, which is in $\Lambda \backslash \check{X}_k \cup \check{Y}_{n-k}$. Note that interaction between $X_R^m \cup \check{Y}_{n-k}$ and $\check{X}_k$ is zero due to finite range of potential. So we have

$$\exp\{-\beta V_{st}(x)_m - \beta V((x)_m;\gamma_\Lambda) - \frac{1}{2}\beta V(\gamma_\Lambda)\}$$
$$= E_1((x)_m \cup \check{Y}_{n-k})\, E_2((x)_m \cup \check{Y}_{n-k}; (\check{X}_k \cup \check{Y}_{n-k})^c)\, E_0(\Lambda \backslash \check{Y}_{n-k}) \tag{4.1}$$

where

$$E_1 = e^{-\beta V_{st}(x)_m} \prod_{l=1}^{n-k} e^{-\beta V((x)_m;\gamma_{Y_l}) - (1/2)\beta V(\gamma_{Y_l})} \tag{4.2}$$

$$E_2 = \prod_{j=1}^{m} e^{-\beta V(x_j;\gamma_{(\check{X}_k \cup \check{Y}_{n-k})^c})} \prod_{l=1}^{n-k} e^{-\beta V(\gamma_{Y_l};\gamma_{(\check{X}_k \cup \check{Y}_{n-k})^c})}, \tag{4.3}$$

and

$$E_0 = e^{-\beta V(\gamma_{\Lambda \backslash \check{Y}_{n-k}})} \tag{4.4}$$

Now splitting the energy in the product of (4.2)

$$V = V_{st} + V_+$$

and taking into account that (due to (A.2))

$$V_{st}(x)_m + \sum_{l=1}^{n-k} V_{st}((x)_m; \gamma_{Y_l}) + \tfrac{1}{2} \sum_{l=1}^{n-k} V_{st}(\gamma_{Y_l})$$
$$= V_{st}((x)_m \cup \gamma_{\tilde{Y}_{n-k}}) \geqslant -B(m + |\gamma_{\tilde{Y}_{n-k}}|) = -B\left(m + \sum_{l=1}^{n-k} \sum_{\Delta \in Y_l} |\gamma_\Delta|\right)$$

(we apply to V_{st} the Eq. (2.6) and neglect the positive parts of the interaction between different elementary cubes) we get

$$E_1 \leqslant e^{m\beta B} \prod_{l=1}^{n-k} \prod_{\Delta \in Y_l} e^{\beta B |\gamma_\Delta| - (1/2)\beta V_+(\gamma_\Delta)} \tag{4.5}$$

Then using the definition (2.7) for v_1 and estimating the exponent of the positive part of energy in (4.3) by the unite we get

$$E_2 \leqslant e^{m\beta v_1} \prod_{l=1}^{n-k} \prod_{\Delta \in Y_l} e^{\beta v_1 |\gamma_\Delta|} \tag{4.6}$$

Now we use the property (2.5) of the measure $\pi_z(d\gamma_\Delta)$ and the estimates (4.5), (4.6) to get that

$$\int \pi_z(d\gamma_{\tilde{Y}_{n-k}})\, \tilde{\chi}_+^{\tilde{Y}_{n-k}}(\gamma_{\tilde{Y}_{n-k}})\, E_1 E_2 \leqslant e^{m\beta(B+v_1)} \prod_{l=1}^{n-k} \prod_{\Delta \in Y_l} I_\Delta$$

where

$$I_\Delta = \int \pi_z(d\gamma_\Delta)\, \chi_+^\Delta(\gamma_\Delta)\, e^{\beta(v_1+B)|\gamma_\Delta| - (1/2)\beta V_+(\gamma_\Delta; \gamma_\Delta)}$$
$$= \sum_{n \geqslant 2} \frac{z^n}{n!} e^{\beta(v_1+B)n} \int_{\Delta^n} dx_1 \cdots dx_n \exp\left\{-\beta \sum_{1 \leqslant i < j \leqslant n} V_+(x_i - x_j)\right\}$$

So as all $(x)_n$ are in the same cube

$$V_+(x_i - x_j) \geqslant b$$

where b is defined in (A3). So, we have

$$I_\Lambda \leqslant \varepsilon_1(z, \beta, \lambda) = \varepsilon_1 \tag{4.7}$$

$$\varepsilon_1 = \tfrac{1}{2} z^2 \lambda^{2\nu} e^{-\beta(1/2b - 2v_1 - 2B)} \exp\{z\lambda^\nu e^{-\beta(3/4b - v_1 - B)}\} \tag{4.8}$$

which is small for small λ and due to (A3).

Now taking maximum of E_0 in the variable $\check{Y}_{n-k}$ (we note this maximum value of $\check{Y}_{n-k}$ by $\bar{Y}_{n-k}$) and using the elementary estimate

$$\chi_R^{cor}((X)_k, (Y)_{n-k}) \leqslant \chi_R^{cor}(X)_k \tag{4.9}$$

we can estimate the sum over $(Y)_{n-k}$. This estimate are given by the following Lemma:

Lemma 4.1. (see ref. 14)

$$\sum_{Y \cap X_R^m \neq \varnothing} \varepsilon_1^{|Y|/|\Delta|} \leqslant mc(\nu)\, R^\nu \lambda^{-\nu} \frac{\varepsilon}{1-\varepsilon} = mK(z, \beta, \lambda, \varphi) = mK \tag{4.10}$$

where $c(\nu)$ is the constant, which depends only on ν, and

$$\varepsilon = 2c(\nu)\, R^\nu z^2 e^{-\beta(1/2b - 2v_1 - 2B)} \exp\{z\lambda^\nu e^{-\beta(3/4b - v_1 - B)}\}\, \lambda^\nu \tag{4.11}$$

The last step is the following. The expansion like (3.8) one can make for partition function Z_{Λ_1} with $\Lambda_1 \subset \Lambda$. Note it by

$$Z_{\Lambda_1} = \sum_{k \geqslant 0} \frac{1}{k!} Z_{\Lambda_1}^{(k)}$$

and noting by

$$Z_{\Lambda \setminus \bar{Y}_*}^{(k)} = \max_{n \geqslant 0} Z_{\Lambda \setminus \bar{Y}_n}^{(k)}$$

and taking into account (4.5), (4.6), (4.8), (4.10) and (4.11) we get

$$\tilde{\rho}^\Lambda(x)_m \leqslant e^{m\beta(B+v_1)} \sum_{n \geqslant 0} \sum_{k=0}^{n} \frac{(mK)^{n-k}}{k!\,(n-k)!} Z_{\Lambda \setminus \bar{Y}_{n-k}}^{(k)} \leqslant Z_{\Lambda \setminus \bar{Y}_*} e^{m(\beta B + \beta v_1 + K)}$$

The facts that $Z_{\Lambda_1} \leqslant Z_{\Lambda_2}$ for $\Lambda_1 \subset \Lambda_2$ and (3.4) give the inequality (2.8) with

$$\xi = z e^{\beta(v_1 + B) + K}$$

Now we prove the lemma.

Proof of Lemma 4.1. First of all, we write that

$$\sum_{Y \cap X_R^n \neq \emptyset} \varepsilon_1^{|y|/|\Delta|} \leqslant \sum_{\Delta \in X_R^m} \sum_{Y,\, \Delta \in Y,\, \Delta - \text{fixed}} \varepsilon^{|Y|/|\Delta|}$$

and

$$\sum_{Y,\, \Delta \in Y,\, \Delta - \text{fixed}} \varepsilon_1^{|y|/|\Delta|} = \sum_{k \geqslant 1} \varepsilon_1^k \sum_{Y,\, \Delta \in Y,\, \Delta - \text{fixed},\, |Y| = \lambda^\nu k} 1$$

Let S_k be the number of all R-connected sets Y with fixed $\Delta \in Y$ and $|Y| = \lambda^\nu k$. Note N_1 is the number of elementary cubes Δ' around fixed cube Δ such that $\operatorname{dist}(\Delta, \Delta') \leqslant R$. It is clear that $N_1 = [c(\nu)(R/\lambda)^\nu]$, where $c(\nu)$ is some constant which depends only on ν, $[\cdot]$-means integer part. Now, if we consider the set T as the set of all elementary cubes Δ and $\Gamma = (A_1,..., A_n)$, $A_j = (\Delta_{j_1}, \Delta_{j_2})(A_j \neq \emptyset \text{ iff } \operatorname{dist}(\Delta_{j_1}, \Delta_{j_2}) \leqslant R)$ as some collection of its subsets, then the proof of the lemma immediately follows from the Lemma 1.3, Chapt. 2 of ref. 14 with $M = 2$, $K = N_1$ and $r = 1/4K = 1/4N_1$, which imply

$$S_k < (4c(\nu)\, R^\nu \lambda^{-\nu})^k$$

and as a consequence the proof of the lemma. ∎

Concluding Remark

The proof of the Main Theorem remains also true if we add to the 2-body interaction $V(x)_N$ an arbitrary finite-range *many-body* potential which satisfies *stability* condition (2.6).

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